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# INVENTORY MANAGEMENT

## EXAM

EBM026A05 2016–2017.2A FEB RuG

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### Read me:

- The exam has a duration of 2 hours.
- The exam has 5 questions, for a total of 20 points.
- The exam is open book. That is, you are allowed to use the textbook Factory Physics (hardcover or print on paper) during the exam, but nothing else.
- We cannot not assign points when your answer is only partly correct, vague, illegibly written, lacks motivation, or not to-the-point.
- You can use the backside of the pages as scrap paper or to continue your answers, if necessary.
- You can also use a calculator that is compliant with the rules as set out by FEB, i.e. no programmable calculators or cell phones.

1. Discuss (very shortly) the following.

- (a) Under what condition (parameter value) would the optimal order quantity for the EOQ model with and without backorders be nearly equal? (1p)

**Solution:** When the backorder costs are very high.

- (b) Under what condition (parameter value) would the optimal order quantity for the EOQ model and the optimal production quantity for the EPQ model be nearly equal? (1p)

**Solution:** When the production rate is very high.

2. It is often the case that suppliers encourage large purchases by using quantity discounts. However, the opposite can also be the case. Consider a company whose supplier offers a quantity premium; if the order quantity is below or equal to a certain threshold  $\bar{Q}$  then the procurement cost (buying price) is  $c_0$ , and if the order quantity is larger than  $\bar{Q}$ , the procurement cost is  $c_1$ , where  $c_1 > c_0$ . This question entails extending the standard EOQ model to account for this case. Therefore, all the assumptions of the standard EOQ model still apply except that a quantity premium, rather than a discount, is in place. The parameter values are like this: demand rate is 50,000 units per year, fixed ordering cost is \$10 per replenishment, carrying charge is  $\alpha = 20\%$  per year (i.e. if the procurement cost is \$1 then the holding cost is  $\alpha h = 0.2 \cdot \$1 = \$0.2$  per year),  $c_0$  is \$1.00 per unit,  $c_1$  is \$1.005 per unit, and  $\bar{Q}$  is 1500 units.

*Note:* Do not forget to consider procurement costs in the following questions.

- (a) What would be the total cost per year if we were to order in quantities  $\bar{Q}$ ? (1p)

**Solution:** This would be

$$f(\bar{Q}) = AD/\bar{Q} + \alpha c_0 \bar{Q}/2 + c_0 D = \frac{10 \cdot 50000}{1500} + \frac{0.2 \cdot 1 \cdot 1500}{2} + 1 \cdot 50000 = 50483.33.$$

- (b) Write an expression for the total cost per year as a function of the order quantity  $Q$ . (2p)

**Solution:** The average cost function in the classical case is  $f(Q) = AD/Q + \alpha cQ/2 + cD$ . We can re-write this as:

$$f(Q) = \begin{cases} AD/Q + \alpha c_0 Q/2 + c_0 D & \text{if } Q \leq \bar{Q} \\ AD/Q + \alpha c_1 Q/2 + c_1 D & \text{if } Q > \bar{Q} \end{cases}$$

- (c) What is the optimal order quantity if we want to buy at a procurement cost of  $c_0$ ? (1p)

**Solution:** In this case, we have

$$Q^* = \sqrt{\frac{2AD}{\alpha c_0}} = \sqrt{\frac{2 \cdot 10 \cdot 50000}{0.2 \cdot 1}} = 2236.07.$$

The optimal quantity  $Q^*$  is larger than  $\bar{Q}$ . Therefore, if we want to buy at a procurement cost of  $c_0$ , the best we can do is to order  $\bar{Q}$ .

- (d) What is the optimal order quantity if we are willing to buy at a procurement cost of  $c_1$ ? (1p)

**Solution:** In this case, we have

$$Q^* = \sqrt{\frac{2AD}{\alpha c_1}} = \sqrt{\frac{2 \cdot 10 \cdot 50000}{0.2 \cdot 1.005}} = 2230.50.$$

The optimal quantity  $Q^*$  is larger than  $\bar{Q}$ . Therefore, it is applicable.

- (e) What is the optimal order quantity? (1p)

**Solution:** We should compare the two cases where the procurement cost is  $c_0$  and  $c_1$ . We know from (c) and (d) the optimal order quantities in these cases are 1500 and 2230.50, respectively.

If we plug these quantities into the expression in (b) we obtain

$$f(\bar{Q}) = AD/\bar{Q} + \alpha c_0 \bar{Q}/2 + c_0 D = \frac{10 \cdot 50000}{1500} + \frac{0.2 \cdot 1 \cdot 1500}{2} + 1 \cdot 50000 = 50483.33$$

for the first case and

$$\begin{aligned} f(Q^*) &= AD/Q^* + \alpha c_1 Q^*/2 + c_1 D = \sqrt{2AD\alpha c_1} + c_1 D \\ &= \sqrt{2 \cdot 10 \cdot 50000 \cdot 0.2 \cdot 1.005} + 1.005 \cdot 50000 = 50698.33 \end{aligned}$$

for the second case.

Comparing the two cases, we conclude that the optimal order quantity is 1500, leading to an annual cost of \$50,483.33.

3. Consider a newsvendor problem with the following parameters; unit procurement cost (buying price)  $c = \$10$ , selling price  $p = \$40$ , and salvage value  $s = \$0$ .

- (a) Assume that the demand  $D$  is discrete and uniformly distributed on  $\{0, 1, \dots, 9\}$ . That is  $\Pr\{D = 0\} = \Pr\{D = 1\} = \dots = \Pr\{D = 9\} = 0.1$ . Compute the optimal order quantity (i.e., the quantity that maximizes the expected profit). (2p)

**Solution:** We first compute the overage cost  $h$  and the shortage cost  $b$ . We have  $h = c - s = 10 - 0 = 10$  and  $b = p - c = 40 - 10 = 30$ . We know that

$$F(Q^*) = \min \left\{ Q \mid F(Q) \geq \frac{b}{h+b} \right\} = \min \{ Q \mid F(Q) \geq 0.75 \}$$

Therefore, we need to find the smallest order quantity such that the probability of not stocking-out is greater than or equal to 75%. The demand is discrete and uniformly distributed on  $\{0, 1, \dots, 9\}$ . Therefore, the distribution function  $f(x)$  and cumulative distribution function  $F(x)$  are as follows.

$x$	0	1	2	3	4	5	6	7	8	9
$f(x)$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
$F(x)$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

It is clear that, in order to attain a non-stockout probability larger than or equal to 75%, we need to order at least 7 units.

- (b) Assume that the demand  $D$  is normally distributed with mean  $\mu = 4.5$  and standard deviation  $\sigma = 1.5$ . Compute the optimal order quantity. (1p)

*Note:* To answer this question, you will need to use the  $z$ -table provided on the last page of the exam (and also in the Factory Physics). The  $z$ -table provides values of the standard normal cumulative distribution function evaluated at different  $z$  values (from 0 to 3).

**Solution:** We once again need to find the minimum order quantity that would yield a non-stockout probability larger than or equal to 75%. We can find this value by using the  $z$ -table. To that end, we first need to find the  $z$  value that yields a cumulative probability of 75%. A quick look at the table shows that this value is some where between 0.67 (with probability 0.7486) and 0.68 (with probability 0.7517). For convenience, we take  $z = 0.68$ . The next step is to find which order quantity this  $z$  value corresponds to. We have that

$$z = \frac{Q - \mu}{\sigma}.$$

Therefore, plugging in  $\mu$  and  $\sigma$  would gives

$$0.68 = \frac{Q - 4.5}{1.5} \rightarrow Q = 5.52.$$

This shows that, in order to attain a non-stockout probability larger than or equal to 75%, we need to order at least 5.52 units.

- (c) Compare the optimal ordering quantities you have obtained for uniformly distributed demand and normally distributed demands. Discuss (very shortly) whether they are different and if so why one is larger than the other. (1p)

**Solution:** The optimal ordering quantity is larger for the uniformly distributed demand as compared to the normally distributed demand. This is due to the extent of uncertainty. The demand is concentrated around the mean 4.5 for the normal distribution whereas it is spread evenly over between 0 and 9 for the uniform distribution.

4. The  $(s, S)$  policy is an inventory control policy where an order is placed to increase the inventory position up to  $S$  whenever the inventory position drops below  $s$ . Note that  $(s, S)$  and  $(Q, r)$  policies approach the “when to order?” question similarly. The analogy between the  $r$  of the  $(Q, r)$  policy and  $s$  of the  $(s, S)$  policy is apparent, both  $s$  and  $r$  act as re-order levels. These two policies, however, approach to the “how much to order?” question quite differently: the  $(Q, r)$  policy uses a fixed order size  $Q$ , whereas the  $(s, S)$  policy uses an order-up-to level  $S$ .

We have a supermarket that uses an  $(s, S)$  policy to control its inventory of a particular item. The lead time  $L$  is positive and constant.

- (a) Formulate a set of recursions to simulate the system over time by making use of the notation provided below. (2p)

$D_n$  the demand during period  $n$

$IP_n$  the inventory position at the end of period  $n$

$IL_n$  the inventory level at the end of period  $n$

$Q_n$  the order quantity in period  $n$

**Solution:** Three recursions are sufficient.

1. The order quantity in period  $n$  is  $S - IP_{n-1}$  if the inventory position at the end of the previous period is smaller than or equal to  $s$ , and zero otherwise (assuming that  $s \leq S$  as it would not make sense otherwise):

$$Q_n = (S - IP_{n-1}) \mathbb{1}\{IP_{n-1} \leq s\}$$

2. The inventory level at the end of period  $n$  equals the inventory level at the end of the previous period plus the orders received in period  $n$  (placed in period  $n - L$ ) minus the demand in period  $n$ :

$$IL_n = IL_{n-1} + Q_{n-L} - D_n$$

3. The inventory position at the end of period  $n$  is the inventory position at the end of the previous period plus the order quantity in period  $n$  minus the demand in period  $n$ :

$$IP_n = IP_{n-1} + Q_n - D_n$$

- (b) By using the recursions you derived, simulate the system over four periods using the following parameters;  $s = 3$ ,  $S = 7$ , and  $L = 1$ . Assume that initial inventory position and inventory level are  $IP_0 = 6$  and  $IL_0 = 6$  (note that there is no outstanding orders at the outset) and demand over four periods are  $D_1 = 4, D_2 = 4, D_3 = 2, D_4 = 1$ . Compute  $IP_n$  and  $IL_n$  as well as order quantities  $Q_n$  for  $n = 1, \dots, 4$ . (1p)

**Solution:** If we plug in the data into recursions; we obtain the following:

$$n = 1: \quad Q_1 = (S - IP_0) \mathbb{1}\{IP_0 \leq s\} = (7 - 6) \mathbb{1}\{6 \leq 3\} = 0$$

$$IL_1 = IL_0 + Q_0 - D_1 = 6 + 0 - 4 = 2$$

$$IP_1 = IP_0 + Q_1 - D_1 = 6 + 0 - 4 = 2$$

$$n = 2: \quad Q_2 = (S - IP_1) \mathbb{1}\{IP_1 \leq s\} = (7 - 2) \mathbb{1}\{2 \leq 3\} = 5$$

$$IL_2 = IL_1 + Q_1 - D_2 = 2 + 0 - 4 = -2$$

$$IP_2 = IP_1 + Q_2 - D_2 = 2 + 5 - 4 = 3$$

$$n = 3: \quad Q_3 = (S - IP_2) \mathbb{1}\{IP_2 \leq s\} = (7 - 3) \mathbb{1}\{3 \leq 3\} = 4$$

$$IL_3 = IL_2 + Q_2 - D_3 = -2 + 5 - 2 = 1$$

$$IP_3 = IP_2 + Q_3 - D_3 = 3 + 4 - 2 = 5$$

$$n = 4: \quad Q_4 = (S - IP_3) \mathbb{1}\{IP_3 \leq s\} = (7 - 5) \mathbb{1}\{5 \leq 3\} = 0$$

$$IL_4 = IL_3 + Q_3 - D_4 = 1 + 4 - 1 = 4$$

$$IP_4 = IP_3 + Q_4 - D_4 = 5 + 0 - 1 = 4$$

In what follows, disregard the data provided and the results obtained in the previous part of the question, and assume that the simulation led to the following  $D_n$ ,  $Q_n$ ,  $IL_n$ , and  $IP_n$  values:

$n$	1	2	3	4
$D_n$	1	5	1	10
$Q_n$	0	0	10	0
$IL_n$	5	0	-1	-1
$IP_n$	5	0	9	-1

*Note:* The values given in this table do not relate the solution of the previous part of the question at all.

- (c) Compute the average on-hand inventory level and average backorders based on the results you obtained in the simulation. (1p)

**Solution:** The on-hand inventory level at the end of period  $n$  equals  $IL_n^+$ . The average over four periods is

$$\frac{\sum_{n=1}^4 IL_n^+}{4} = \frac{5 + 0 + 0 + 0}{4} = 1.25.$$

The backorder level at the end of period  $n$  equals  $IL_n^-$ . The average over four periods is

$$\frac{\sum_{n=1}^4 IL_n^-}{4} = \frac{0 + 0 + 1 + 1}{4} = 0.5.$$

- (d) Compute the  $\alpha$ -service level (non-stockout probability) of the system based on your simulation results. (1p)

**Solution:** The  $\alpha$ -service level is the probability of not stocking out. In our simulation experiment we have four periods, therefore we have four observations on stocking-out or not stocking out. The non-stockout probability can thus be computed as

$$\frac{\sum_{n=1}^4 \mathbb{1}\{IL_n \geq 0\}}{4} = \frac{1 + 1 + 0 + 0}{4} = 0.5.$$

- (e) Compute the  $\beta$ -service level (fill rate) of the system based on your simulation results. (1p)

**Solution:** The  $\beta$ -service level is one minus the ratio of the expected shortage and the expected demand. In our simulation experiment we have four periods, therefore we have four observations on shortages and demands.

The expected shortage is

$$\begin{aligned} \frac{\sum_{n=1}^4 \min\{IL_n^-, D_n\}}{4} &= \frac{\min\{0, 1\} + \min\{0, 5\} + \min\{1, 1\} + \min\{1, 10\}}{4} \\ &= \frac{0 + 0 + 1 + 1}{4} = 0.5. \end{aligned}$$

*Note:* Your answer will also be considered correct if – just like we did in the mock exam – you confused the expected backorder with the expected shortage, and used the following expression for expected shortage (which in this case gives the same result)

$$\frac{\sum_{n=1}^4 IL_n^-}{4} = \frac{0 + 0 + 1 + 1}{4} = 0.5.$$

The expected demand is

$$\frac{\sum_{n=1}^4 D_n}{4} = \frac{1 + 5 + 1 + 10}{4} = 4.25.$$

Therefore,  $\beta$ -service level is  $1 - \frac{0.5}{4.25} = 0.882$ .

5. Consider an inventory system with two products with deterministic demands. The demands are  $D_1 = 1000$  and  $D_2 = 10$  per year. The holding costs are  $h_1 = \$1$  and  $h_2 = \$100$  per year. There is no individual fixed ordering cost, but a joint fixed ordering cost  $A = \$50$ .

- (a) The inventory manager believes that it makes sense to order both items whenever a replenishment order is placed, because there is no individual fixed ordering cost. Discuss (very shortly) whether you agree with the inventory manager. (1p)

**Solution:** The manager is indeed right. Because there is no individual fixed ordering costs, the individual cost is comprised solely of the holding cost. Therefore, one can only reduce the individual cost by reducing average inventory level – which is possible by reducing the order quantities and increasing order frequency.

- (b) Suppose that the inventory manager is right. That is, we use the same replenishment interval (review period) for both products. Find a replenishment interval and corresponding order quantities that minimize the total cost per year using a power-of-two policy. Make sure that your order interval is a power-of-two multiple of a day. (1p)

**Solution:** For a given replenishment interval  $T = 2^n$  (in days); the order quantities are  $Q_1 = \frac{D_1 2^n}{365}$  and  $Q_2 = \frac{D_2 2^n}{365}$ . The holding costs per year are therefore  $\frac{h_1 Q_1 + h_2 Q_2}{2}$  and the fixed replenishment cost per year is  $\frac{A \cdot 365}{2^n}$ . Thus, the total cost (as a function of  $n$ ) can be written as

$$C(n) = \frac{A \cdot 365}{2^n} + \frac{(h_1 D_1 + h_2 D_2) \cdot 2^n}{2 \cdot 365}$$

$$= \frac{50 \cdot 365}{2^n} + \frac{(1 \cdot 1000 + 100 \cdot 10) \cdot 2^n}{2 \cdot 365} = \frac{50 \cdot 365}{2^n} + \frac{1000 \cdot 2^n}{365}.$$

If we start from zero and compute this expression for increasing values of  $n$  as long as  $C(n)$  is decreasing, we obtain the following results which suggest that the optimal power-of-two replenishment interval is 64 days.

$n$	$2^n$	$C(n)$
0	2	9130.48
1	4	4573.46
2	8	2303.17
3	16	1184.46
4	32	657.98
5	64	<b>460.50</b>
6	128	493.26

$z$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990